BOUNDS ON INTEGRALS WITH APPLICATIONS TO RELIABILITY PROBLEMS

by

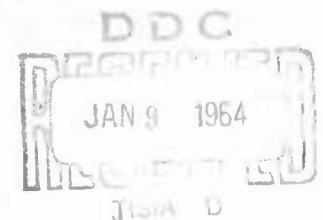Richard E. Barlow

OPERATIONS RESEARCH CENTER

INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA - BERKELEY

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Richard E. Barlow
Operations Research Center
University of California, Berkeley

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1. Summary

Certain statistical problems in life testing and reliability lead to the consideration of minimizing and maximizing certain integrals of the form

$$\int_0^\infty \phi\,(x\,,\,\overline{F}(x))\;dx \qquad \overline{F}(x) = 1 - F(x)$$

under the assumption that the distribution function F has increasing hazard rate and that one or two moments are given. The minimizing (maximizing) distributions for the special cases considered are members of the extremal class considered by A. Marshall and the author [1]. Bounds on the expected values of the minimum and maximum order statistics as well as the operating characteristics of replacement policies based on age are obtained under these assumptions.

2. Statistical Motivation

For many statistical problems in life testing and reliability it is natural to assume that the life distribution F $(F(0^-) = 0)$ has increasing hazard rate (is IHR). If F has density f , then F is IHR if and only if $f(x)/[1 - F(x)]$ is nondecreasing in x . In general F is said to be IHR if and only if $\ln[1 - F(x)]$ is concave where finite. This property is enjoyed by most of the commonly used life distributions; e.g.,

$$\text{Weibull:}\quad f(x) = \begin{cases} \lambda\alpha x^{\alpha-1}\exp(-\lambda x^\alpha) & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

for $\alpha \geq 1$, $\lambda > 0$, and

$$\text{Gamma:} \quad f(x) = \begin{cases} \dfrac{\lambda^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\lambda x) & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

for $\alpha \geq 1$, $\lambda > 0$. Distributions F for which $\ln[1 - F(x)]$ is convex on $[0 , \infty)$ are called DHR for decreasing hazard rate. Properties of distributions with monotone hazard rate are discussed in [2].

In addition to this assumption, some prior information in the form of the mean μ_1 , and the variance $\sigma^2 = \mu_2 - \mu_1^2$ of the distribution may be known. Since in the IHR case

$$\frac{\sigma}{\mu_1} \leq 1$$

with equality only for exponential distribution, an estimate of the coefficient of variation σ/μ_1 may represent the statistician's belief in the amount of departure of the failure distribution from exponentiality.

With the mean and variance assumed known we consider the problem of minimizing and maximizing certain integrals of the form

$$(2.1) \qquad \int_0^\infty \phi \ (x \ , \ \overline{F}(x)) \ dx$$

where F is IHR. Integrals of such functions have been considered by Karlin [3], Rustagi [4] and others for general distributions. (See [4] for additional references). A related but more specialized problem is that of minimizing (maximizing) integrals of the form

$$\int_0^\infty \rho(x) \ dF(x)$$

assuming that certain moments of F are specified, (cf. (3.1) and Theorem 4.1). Assuming no conditions other than the moment conditions, this problem has been solved by H. P. Mulholland and C. A. Rogers (1958). Their methods as well as those of many others (e.g. Karlin and Shapley (1953) use the fact that the class of all distributions subject to specified moment conditions are closed under convex combinations. However, the class of IHR distributions is not closed under convex combinations [2, p. 381] and, therefore, we cannot use the classical methods. Instead we consider very special cases using the methods of [1]. However, we do obtain fairly explicit bounds which can be easily machine computed.

Integrals such as (2.1) often arise in statistical applications. For example, let $U_1 \leq U_2 \leq \ldots \leq U_n$ denote n ordered observations from a population with IHR cdf F. Let $W_n = U_n - U_1$ denote the sample range. Then we may seek to minimize (maximize)

$$E(W_n) = \int_0^\infty [1 - F(x)^n - (\overline{F}(x))^n] \, dx$$

where $\overline{F}(x) = 1 - F(x)$. The integrand is of the form $\phi(x, y)$ where

$$\phi(x, y) = 1 - (1 - y)^n - y^n$$

is concave in y.

For another example, consider a replacement policy of the form: Replace at time t or at failure, whichever occurs first. We may then seek to minimize (maximize) the following integrals:

$$\text{(i)} \quad \int_0^t \overline{F}(x) \, dx \qquad \overline{F}(x) = 1 - F(x)$$

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i.e., the expected time between removals for either a failed or nonfailed item.

$$(ii) \qquad \frac{\int_0^t \overline{F}(x) \ dx}{F(t)}$$

i.e., the expected time between failure removals using a replacement policy based on t . The reciprocal of this quantity occurs as an upper bound on the renewal function divided by t , [5].

$$(iii) \qquad \frac{\int_0^t \overline{F}(x) \ dx}{\overline{F}(t)}$$

i.e., the expected time between planned replacements using a replacement policy based on t .

A related expression is:

$$(iv) \qquad \int_t^\infty \overline{F}(x) \ dx \Big/ \overline{F}(t)$$

i.e., the mean residual life of an item aged t . Bounds on densities and failure rate functions will be discussed in a future joint paper with A. Marshall.

3. Bounds on Integrals when F is IHR with Specified Mean

If $\phi(x , y)$ is sufficiently smooth the problem of minimizing (maximizing)

$$\int_0^\infty \phi(x , \overline{F}(x)) \ dx$$

is comparatively simple. If F is IHR with mean μ_1 and $\rho(x)$

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is increasing (decreasing) then

$$(3.1) \quad \int_0^{\mu_1} \rho(x) \, dx \underset{(\geq)}{\leq} \int_0^\infty \rho(x) \, \overline{F}(x) \, dx \underset{(\geq)}{\leq} \int_0^\infty \rho(x) \, e^{\dfrac{x}{-\mu_1}} \, dx$$

This result is essentially due to Karlin and Novikoff and follows easily from their method of proof in [6]. Using (3.1) and the method of proof in Karlin [3] and Rustagi [4] we obtain

THEOREM 3.1: If

    (i)   F is IHR , $F(0) = 0$,

    (ii) $\displaystyle\int_0^\infty x \, dF(x) = \mu_1$,

    (iii) $\phi(x , y)$ is convex (concave) in y

    (iv) $\dfrac{\partial}{\partial y} \phi(x , y)$ is nonincreasing (nondecreasing) in x ,

then

$$\int_0^\infty \phi(x , \overline{G}_{\mu_1}(x)) \, dx \underset{(\leq)}{\geq} \int_0^\infty \phi(x , \overline{F}(x)) \, dx \underset{(\leq)}{\geq} \int_0^\infty \phi(x , e^{\dfrac{x}{-\mu_1}}) \, dx$$

when the indicated integrals exist and where G_{μ_1} is the distribution degenerate at μ_1 .

PROOF: We show the lower bound, assuming $\phi(x , y)$ is convex in y and $\dfrac{\partial}{\partial y} \phi(x , y)$ nonincreasing in x . Let $\overline{G}_0(x) = e^{-x/\mu_1}$ and define

$$(3.2) \quad I(\lambda) = \int_0^\infty \phi \, [x , \lambda \overline{G}_0(x) + (1-\lambda) \, \overline{F}(x)] \, dx \quad \text{for } 0 \leq \lambda \leq 1 .$$

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Using (iii) it is easy to check that $I(\lambda)$ is convex in λ . If $I(\lambda)$ achieves its minimum at $\lambda = 1$ then certainly $\overline{G}_o(x)$ will minimize (2.1). This is possible if and only if

$$I'(\lambda)\Big|_{\lambda = 1} \leq 0 \; .$$

Now

$$I'(\lambda)\Big|_{\lambda=1} = \int_0^\infty \frac{\partial}{\partial y} \phi \; [x \; , \; \overline{G}_o(x)][\overline{G}_o(x) - \overline{F}(x)] \; dx.$$

By (iv) $\frac{\partial}{\partial y} \phi(x \; , \; y)$ is nonincreasing in x . By assumption (iii) $\frac{\partial}{\partial y} \phi(x \; , \; y)$ is nondecreasing in y . But if we let

$y = \overline{G}_o(x)$, then

$\frac{\partial}{\partial y} \phi(x \; , \; y)\Big|_{y=\overline{G}_o(x)}$ is nonincreasing in x .

Therefore, by (3.1)

$$\int_0^\infty \frac{\partial}{\partial y} \phi(x \; , \; y)\Big|_{y=\overline{G}_o(x)} \overline{F}(x) \; dx$$

$$\geq \quad \int_0^\infty \frac{\partial}{\partial y} \phi(x \; , \; y)\Big|_{y=\overline{G}_o(x)} \overline{G}_o(x) \; dx$$

and hence $I'(\lambda)\Big|_{\lambda=1} \leq 0$. Therefore

$$\int_0^\infty \phi \; (x \; , \; \overline{F}(x)) \; dx \geq \int_0^\infty \phi \; (x \; , \; \overline{G}_o(x)) \; dx \; .$$

If we assume $\phi \; (x \; , \; y)$ is concave in y and $\frac{\partial}{\partial y} \phi \; (x \; , \; y)$ is nondecreasing in x , then all inequalities are reversed.

The other inequalities follow in a similar manner. $\parallel$

Of course an analogous theorem can be stated for DHR distributions.

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Let X_1 , $X_2, \ldots, X_n$ denote n independent observations on a random variable X with IHR distribution F . Let $U_1 \leq U_2 \leq \cdots \leq U_n$ denote the associated order statistics. Then

$$E[U_n - U_1] = \int_0^\infty \{1 - [\overline{F}(x)]^n - [1 - \overline{F}(x)]^n\} \, dx$$

$$= \int_0^\infty \emptyset \, (x \, , \, \overline{F}(x)) \, dx$$

where $\emptyset \, (x \, , \, y) = 1 - y^n - (1 - y)^n$. Since $\emptyset$ is concave in y, Theorem 3.1 applies and we have

$$0 \leq E[U_n - U_1] \leq \mu_1 \sum_{k=1}^{n-1} 1/k$$

for sharp upper and lower bounds on the expected range. Intuitively, this inequality tells us that observations on IHR random variables tend to cluster together more closely than observations from an exponentially distributed random variable with the same mean. In a similar manner we obtain sharp upper and lower bounds on the expected values of the minimum and the maximum observations.

$$(3.2) \qquad \frac{\mu_1}{n} \leq E[U_1] = \int_0^\infty [\overline{F}(x)]^n \, dx \leq \mu_1$$

$$(3.3) \qquad \mu_1 \leq E[U_n] = \int_0^\infty [1 - F^n(x)] \, dx \leq \mu_1 \sum_1^n 1/k$$

Note that $\emptyset \, (x \, , \, y) = \alpha x^{\alpha-1}[1 - (1 - y)^n]$ satisfies the hypotheses of Theorem 3.1 for $\emptyset \, (x \, , \, y)$ concave in y and $\frac{\partial}{\partial y} \emptyset(x \, , \, y)$ non-decreasing in x when $\alpha \geq 1$. Using this $\emptyset$ we can bound all moments of U_n . Namely

$$\int_0^\infty \alpha x^{\alpha-1} [1 - G_{\mu_1}^n(x)] \, dx \le E[U_n^\alpha]$$

$$= \int_0^\infty \alpha x^{\alpha-1} [1 - F^n(x)] \, dx \le \int_0^\infty \alpha x^{\alpha-1} [1 - G_0^n(x)] \, dx.$$

Inequalities (3.2) and (3.3) can be extended to include non-identically distributed random variables. The following proof is due to A. Marshall.

THEOREM 3.2: If $F_i(x)$ is IHR (DHR) with mean μ_i and $\bar{G}_i(x) = e^{-x/\mu_i}$ $(i = 1, 2,\ldots,n)$, then

$$\text{(i)} \quad \int_0^\infty \prod_{i=1}^n \bar{F}_i(x) \, dx \underset{(\le)}{\ge} 1/\sum_1^n 1/\mu_i$$

$$\text{(ii)} \quad \int_0^\infty [1 - \prod_{i=1}^n F_i(x)] \, dx \underset{(\ge)}{\le} \int_0^\infty [1 - \prod_{i=1}^n G_i(x)] \, dx$$

PROOF: First we show (i). By (3.1)

$$\int_0^\infty [\prod_{j=1}^{i-1} \bar{F}_j(x) \prod_{j=i+1}^n \bar{G}_j(x)] \bar{F}_i(x) \, dx$$

$$\underset{(\le)}{\ge} \int_0^\infty [\prod_{j=1}^{i-1} \bar{F}_j(x) \prod_{j=i+1}^n \bar{G}_j(x)] \bar{G}_i(x) \, dx$$

for $1 \le i \le n$. By recursion we obtain

$$\int_0^\infty \prod_{j=1}^n \bar{F}_j(x) \, dx \underset{(\le)}{\ge} \int_0^\infty \prod_{j=1}^n \bar{G}_j(x) \, dx = 1/\sum_1^n \frac{1}{\mu_i}$$

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To show (ii), note that again by (3.1)

$$\int_0^\infty \prod_{j=1}^{i-1} F_j(x) \prod_{j=i+1}^{n} G_j(x) \, \overline{F}_i(x) \, dx$$

$$\underset{(\geq)}{\leq} \int_0^\infty \prod_{j=1}^{i-1} F_j(x) \prod_{j=i+1}^{n} G_j(x) \, \overline{G}_i(x) \, dx .$$

which in turn implies

$$\int_0^\infty [1 - \prod_{j=1}^{i} F_j(x) \prod_{j=i+1}^{n} G_j(x)] \, dx$$

$$\underset{(\geq)}{\leq} \int_0^\infty [1 - \prod_{j=1}^{i-1} F_j(x) \prod_{j=i}^{n} G_j(x)] \, dx .$$

(ii) follows by recursion. ‖

Part (i) tells us that the mean life of a series system with IHR components whose means are μ_i (i=1, 2,..., n) exceeds the mean life of a series system with exponentially distributed components and means μ_i (i=1, 2,..., n). However, just the reverse is true for parallel systems.

Additional properties of order statistics from IHR (DHR) distributions together with applications to life testing will appear in a future report.

4. Bounds on Integrals when F is IHR with Specified Mean and Variance

Assume now that F is IHR with mean μ_1 and μ_2 specified. In [1] the class of extremal distributions for bounding F were determined. Two rather special members of this class are

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$$1 - G_{T_1}(x) = \begin{cases} \exp\,(-\,a_0 x) & x < T_1 \\ 0 & x \geq T_1 \end{cases}$$

and

$$1 - G_{T_0}(x) = \begin{cases} 1 & x \leq 1 - \sqrt{\mu_2 - 1} = T_0 \\ \exp\,\left[\dfrac{-(x - T_0)}{\sqrt{\mu_2 - 1}}\right], & x \geq T_0 \, , \end{cases}$$

where a_0 and T_1 are chosen to satisfy the moment conditions. With these distributions and different assumptions on $\rho(x)$ we can improve the bounds in (3.1). The method we now use has been employed by Karlin and Novikoff [6] among others.

THEOREM 4.1 If

      (i) F is IHR , $F(0) = 0$

      (ii) $\mu_1 = 1$ and μ_2 are specified

      (iii) $\rho(x)$ is convex (concave)

then

$$\int_0^\infty \rho(x)\,\overline{G}_{T_1}(x)\,dx \underset{(\geq)}{\leq} \int_0^\infty \rho(x)\,\overline{F}(x)\,dx$$

$$\underset{(\geq)}{\leq} \int_0^\infty \rho(x)\,\overline{G}_{T_0}(x)\,dx \, .$$

PROOF: Assume $\rho(x)$ is convex. We know [1] that $\overline{F}(x) - \overline{G}_{T_0}(x)$ has exactly two sign changes, say at x_1 and x_2. Furthermore, the order of the sign changes is $-\,+\,-$. Define a line $\ell(x)$ so that

$$\rho(x_1) = \ell(x_1) \quad \text{and} \quad \rho(x_2) = \ell(x_2) \, .$$

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Since ρ is convex, $\rho(x) - \ell(x)$ changes sign at x_1 and x_2 in the order $+ - +$. Therefore,

$$[\rho(x) - \ell(x)][\overline{F}(x) - \overline{G}_{T_0}(x)] \leq 0$$

for all x. Integrating on x we obtain the upper bound on

$$\int_0^\infty \rho(x)\, \overline{F}(x)\, dx \; .$$

The lower bound follows in a similar way using $\overline{G}_{T_1}(x)$ instead of $\overline{G}_{T_0}(x)$.

The ordering of sign changes for $\rho(x) - \ell(x)$ is of course reversed when ρ is concave. $\|$

Using Theorem 4.1 and letting $\rho(x) = x^r$ for $r > 0$, one can obtain improved bounds on μ_r $(r > 2)$ in terms of μ_1 and μ_2.

The general problem of bounding integrals of the form

$$\int_0^\infty \phi\,(x\,,\,\overline{F}(x))\, dx$$

when F is IHR with μ_1 and μ_2 specified seems quite difficult. However, it is possible to treat many special cases using the properties of the extremal family considered in [1]. In order to do this we repeat some of the notation in [1]. We can assume without loss of generality that $\mu_1 = 1$. It follows that μ_2 satisfies $1 \leq \mu_2 \leq 2$. Let

$$T_0 = 1 - \sqrt{\mu_2 - 1} \quad \text{and} \quad T_1 = -a_0^{-1} \log (1 - a_0)$$

where a_0 in $[0\,,\,1]$ and T_1 simultaneously satisfy the following equations:

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$$\int_0^{T_1} e^{-a_0 x} \, dx = \mu_1 = 1$$

$$\int_0^{T_1} x e^{-a_0 x} \, dx = \frac{\mu_2}{2} \ .$$

Let

$$\mathcal{D}_3 = \{G_T : T \geq T_1\}$$

where

$$1 - G_T(x) = \begin{cases} 1 & x < \Delta \\ e^{-a(x-\Delta)} & \Delta \leq x \leq T \qquad T \geq T_1 \\ 0 & x \geq T \end{cases}$$

and a, Δ $(0 \leq \Delta \leq T_0)$ are chosen to satisfy the moment conditions; i.e.,

$$\int_0^{\infty} [1 - G_T(x)] \, dx = \mu_1 = 1$$

$$\int_0^{\infty} x[1 - G_T(x)] \, dx = \frac{\mu_2}{2} \ .$$

Let

$$\mathcal{D}_4 = \{G_T : T_0 \leq T \leq T_1\}$$

where

$$1 - G_T(x) = \begin{cases} e^{-a_1 x} & x \leq T \\ e^{-a_1 T - a_2 (x - T)} & x \geq T \end{cases} \qquad T_0 \leq T \leq T_1$$

and $a_1 \leq a_2$ are chosen to satisfy the moment conditions as

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before. It is shown in [1] that for $t \geq 0$

$$\inf [1 - G_T(t)] \leq 1 - F(t) \leq \sup [1 - G_T(t)]$$

where the extremums are taken over $\mathcal{A}_3 \cup \mathcal{A}_4$. These bounds have
been tabulated for selected values of μ_2 $(1 \leq \mu_2 \leq 2)$.

4.1 Expected Time Between Removals. If X has distribution F ,
then $E[\min (X , t)] =$

$$\int_0^t \overline{F}(x) \, dx .$$

For an age replacement policy this denotes the expected time be-
tween removals using a policy determined by t .

THEOREM 4.2. If F is IHR , $F(0) = 0$ and $\mu_1 = 1$, μ_2
are specified, then

(i) $\qquad \displaystyle\int_0^t \overline{F}(x) \, dx \quad \geq \quad \inf_{G_T \in \mathcal{A}_3} \int_0^t \overline{G}_T(x) \, dx$

(ii) $\qquad \displaystyle\int_0^t \overline{F}(x) \, dx \quad \leq \quad \begin{cases} t & t \leq T_0 = 1 - \sqrt{\mu_2 - 1} \\[2mm] \displaystyle\inf_{G_T \in \mathcal{A}_4} \int_0^t \overline{G}_T(x) \, dx & T_0 \leq t \leq T_1 \\[2mm] \mu_1 = 1 & t \geq T_1 \end{cases}$

All inequalities are sharp.

PROOF: Let $F \in \mathcal{F} - (\mathcal{A}_3 \cup \mathcal{A}_4)$ where $\mathcal{F}$ is the family of all
IHR distributions with prescribed moments $\mu_1 = 1$ and μ_2 .

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Let $T > T_1$ and $s(T)$ be the crossing in (Δ, T) from above of $1 - G_T$ by $1 - F$. We use crucially the fact that $s(T)$ is continuous in T (see [1] p. 52).

Case 1. $t \leq s(T_1)$. From the definition of G_{T_1} we can see that

$$\overline{F}(x) \geq \overline{G}_{T_1}(x) \quad \text{for} \quad 0 \leq x \leq s(T_1)$$

and hence

$$\int_0^t \overline{F}(x) \, dx \geq \int_0^t \overline{G}_{T_1}(x) \, dx .$$

Case 2. $t \geq s(\infty)$. $\overline{F}(x)$ crosses $\overline{G}_{T_0}(x)$ from above at $s(\infty)$ and

$$\overline{F}(x) \leq \overline{G}_{T_0}(x) \quad \text{for} \quad x > s(\infty) .$$

Hence

$$\int_t^\infty \overline{F}(x) \, dx \leq \int_t^\infty \overline{G}_{T_0}(x) \, dx$$

implies

$$\int_0^t \overline{F}(x) \, dx \geq \int_0^t \overline{G}_{T_0}(x) \, dx .$$

Case 3. $s(T_1) < t < s(\infty)$. Since $s(T)$ is continuous in T, choose $T \geq T_1$ and Δ $(0 \leq \Delta \leq T_0)$ such that $s(T) = t$. In this circumstance $G_T \in \mathcal{D}_3$. A drawing will help.

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Let
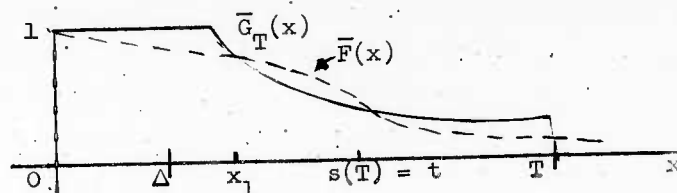
$$\phi_t(x) = \begin{cases} 1 & x \leq t \\ 0 & x > t \end{cases}$$

and define a line $\ell(x)$ such that $\ell(x_1) = 1$ and $\ell(T) = 0$.
Since clearly

$$[\phi_t(x) - \ell(x)][\overline{F}(x) - \overline{G}_T(x)] \geq 0$$

for all x,

$$\int_0^{\infty} [\phi_t(x) - \ell(x)][\overline{F}(x) - \overline{G}_T(x)] \, dx =$$

$$\int_0^t \overline{F}(x) \, dx - \int_0^t \overline{G}_T(x) \, dx \geq 0 .$$

This proves (i).

To prove (ii) we first note that

$$\overline{F}(x) \leq \overline{G}_{T_0}(x) = 1 \qquad \text{for} \qquad 0 \leq x \leq T_0$$

and hence

$$\int_0^t \overline{F}(x) \, dx \leq t \qquad \text{for} \qquad t \leq T_0 ,$$

is sharp.

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Also,

$$\overline{F}(x) \geq \overline{G}_{T_1}(x) \qquad \text{for} \quad x \geq T_1$$

implies

$$\int_0^t \overline{F}(x) \, dx \leq \int_0^t \overline{G}_{T_1}(x) \, dx = \mu_1 = 1 \, . \qquad t \geq T_1$$

is sharp. To complete the proof we need the function $u(T)$ defined for $T_0 \leq T \leq T_1$. Let $u(T)$ be the crossing in $[T, \infty]$ from below of $\overline{G}_T$ by $\overline{F}$; $u(T)$ always exists (see [1]). We use crucially the fact that $u(T)$ varies continuously from $u(T_0) > T_0$ to $u(T_1) = T_1$.

Case 1. $t \leq u(T_0)$. Clearly $\overline{F}(x) \leq \overline{G}_{T_0}(x)$ for $x \leq u(T_0)$ implies

$$\int_0^t \overline{F}(x) \, dx \leq \int_0^t \overline{G}_{T_0}(x) \, dx.$$

Case 2. $T_0 \leq u(T_0) < t < u(T_1) = T_1$. By continuity of $u(T)$ we can choose T such that $u(T) = t$. In this circumstance $G_T \in \mathcal{V}_4$. The following drawing will help.
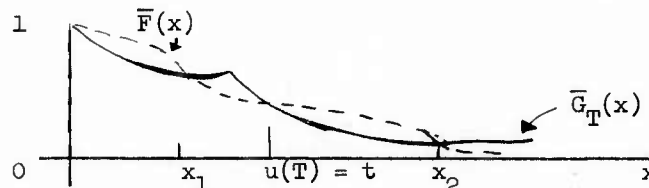
If $\overline{F}(x) \leq \overline{G}_T(x)$ for $x \leq t$, the proof is obvious. Hence suppose $\overline{F}(x)$ crosses $\overline{G}_T(x)$ from above at x_1. If $x_2 = \infty$,

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the proof is also obvious. Hence suppose $x_2 < \infty$. Let

$$\phi_t(x) = \begin{cases} 1 & x \leq t \\ 0 & x > t \end{cases}$$

and construct a line $\ell(x)$ so that $\ell(x_1) = 1$ and $\ell(x_2) = 0$.
Since

$$[\phi_t(x) - \ell(x)][\overline{F}(x) - \overline{G}_T(x)] \leq 0$$

for all x we have

$$\int_0^\infty [\phi_t(x) - \ell(x)][\overline{F}(x) - \overline{G}_T(x)] \, dx$$

$$= \int_0^t \overline{F}(x) \, dx - \int_0^t \overline{G}_T(x) \, dx \leq 0 . \parallel$$

Although the above bounds must be computed numerically, they
can be easily programmed because of the simple form of the
extremal distributions.

If μ_1 is the only moment specified (F IHR) , then

$$\int_0^t \overline{F}(x) \, dx \geq \mu_1 [1 - e^{-t/\mu_1}]$$

$$\int_0^t \overline{F}(x) \, dx \leq \begin{cases} t & t < \mu_1 \\ \mu_1 & t \geq \mu_1 \end{cases} .$$

All inequalities are sharp. The lower bound follows from the
fact that $\overline{F}(x)$ crosses e^{-x/μ_1} at most once and from above.
The inequality is clear for small t . If it were ever violated,
then a fortiori it would also be violated for $t = +\infty$ which is
impossible.

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4.2 Mean Residual Life

The mean residual life of an item aged t is

$$\int_t^\infty \overline{F}(x)\ dx \Big/ \overline{F}(t)\ .$$

If F is IHR , this ratio is necessarily nonincreasing in t .

THEOREM 4.3: If F is IHR , $F(0) = 0$ and $\mu_1 = 1$, μ_2 are specified then

(i)
$$\frac{\int_t^\infty \overline{F}(x)\ dx}{\overline{F}(t)} \geq \begin{cases} \mu_1 - t & t \leq T_0 \\[2ex] \underset{G_T \in \mathscr{Y}_4}{\inf} \dfrac{\int_t^\infty \overline{G}_T(x)\ dx}{\overline{G}_T(t)} & T_0 \leq t \leq T_1 \\[2ex] 0 & t > T_1 \end{cases}$$

(ii)
$$\frac{\int_t^\infty \overline{F}(x)\ dx}{\overline{F}(t)} \leq \begin{cases} \mu_1 = 1 & t = 0 \\[2ex] \underset{G_T \in \mathscr{Y}_3}{\sup} \dfrac{\int_t^\infty \overline{G}_T(x)\ dx}{\overline{G}_T(t)} & 0 < t < \infty \\[2ex] \sqrt{\mu_2 - 1} & t = +\infty \end{cases}$$

All inequalities are sharp.

PROOF: (i) Case 1. $t \leq T_0$. Since $\overline{F}(x) \leq \overline{G}_{T_0}(x)$ for $0 \leq x \leq T_0$ we see that

$$\int_t^\infty \overline{F}(x)\ dx \geq \int_t^\infty \overline{G}_{T_0}(x)\ dx = \mu_1 - t$$

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is sharp and

$$\overline{F}(t) \leq \overline{G}_{T_0}(t) = 1$$

implies

$$\frac{\displaystyle\int_t^\infty \overline{F}(x)\ dx}{\overline{F}(t)} \geq \mu_1 - t \ .$$

Case 2. $T_0 \leq t \leq T_1$. Define $u(T)$ as in Theorem 4.2 . If $t \leq u(T_0)$

$$\frac{\displaystyle\int_t^\infty \overline{F}(x)\ dx}{\overline{F}(t)} \geq \frac{\displaystyle\int_t^\infty \overline{G}_{T_0}(x)\ dx}{\overline{G}_{T_0}(t)} \ .$$

If $u(T_0) < t \leq u(T_1) = T_1$, then choose T and $G_T \in \mathcal{D}_4$ so that $G_T(t) = F(t)$. This can be done since $u(T)$ is continuous in T . By (ii) of Theorem 4.2 we have

$$\int_0^t \overline{F}(x)\ dx \leq \int_0^t \overline{G}_T(x)\ dx \ .$$

Hence

$$\frac{\displaystyle\int_t^\infty \overline{F}(x)\ dx}{\overline{F}(t)} \geq \frac{\displaystyle\int_t^\infty \overline{G}_T(x)\ dx}{\overline{G}_T(t)}$$

as desired.

Case 3. $t > T_1$. We can always choose $T = t$ and $G_T \in \mathcal{D}_3$ so that

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$$\overline{G}_T(x) = 0 \quad \text{for} \quad x > t \quad \text{and} \quad \overline{G}_T(t) \neq 0 .$$

Hence
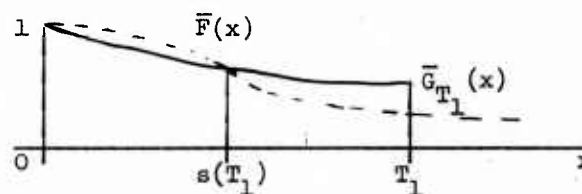
$$\frac{\displaystyle\int_t^\infty \overline{F}(x) \, dx}{\overline{F}(t)} \geq 0$$

for $t > T_1$ is sharp.

(ii) Again we take cases.

<u>Case 1</u>. $t < s(T_1)$. The following drawing illustrates the situation.

Clearly

$$\int_t^\infty \overline{F}(x) \, dx \leq \int_t^\infty \overline{G}_{T_1}(x) \, dx$$

in this case and

$$\overline{F}(t) \geq \overline{G}_{T_1}(t)$$

implies

$$\frac{\displaystyle\int_t^\infty \overline{F}(x) \, dx}{\overline{F}(t)} \leq \frac{\displaystyle\int_t^\infty \overline{G}_{T_1}(x) \, dx}{\overline{G}_{T_1}(t)} .$$
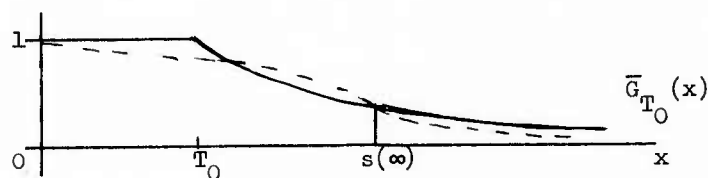
-20-

<u>Case 2</u>. $s(T_1) < t \leq s(\infty)$. By continuity of $s(T)$, there exists
$T \geq T_1$ and Δ such that $s(T) = t$ and $\overline{F}(t) = \overline{G}_T(t)$ $(G_T \in \mathcal{Y}_3)$.
By case 3 (i) of Theorem 4.2

$$\int_0^t \overline{F}(x) \ dx \ \geq \int_0^t \overline{G}_T(x) \ dx$$

implies

$$\frac{\int_t^\infty \overline{F}(x) \ dx}{\overline{F}(t)} \leq \frac{\int_t^\infty \overline{G}_T(x) \ dx}{\overline{G}_T(t)} \ .$$

<u>Case 3</u>. $t \geq s(\infty)$. If $t = s(\infty)$, the following situation holds.

Hence

$$\frac{\int_t^\infty \overline{F}(x) \ dx}{\overline{F}(t)} \leq \frac{\int_t^\infty \overline{G}_{T_0}(x) \ dx}{\overline{G}_{T_0}(t)} = \sqrt{\mu_2 - 1}$$

since

$$\overline{F}(t) = \overline{G}_{T_0}(t) \quad \text{at} \quad t = s(\infty) \ .$$

Since

$$\frac{\int_t^\infty \overline{F}(x) \ dx}{\overline{F}(t)}$$

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is nonincreasing in t we have that

$$\frac{\int_t^\infty \overline{F}(x) \, dx}{\overline{F}(t)} \leq \sqrt{\mu_2 - 1}$$

for $t \geq s(\infty)$ is sharp . ‖

It is easy to show that if only μ_1 is specified (F IHR) ,
then the following inequalities are sharp.

$$\frac{\int_t^\infty \overline{F}(x) \, dx}{\overline{F}(t)} \geq \begin{cases} \mu_1 - t & t \leq \mu_1 \\ 0 & t > \mu_1 \end{cases}$$

$$\frac{\int_t^\infty \overline{F}(x) \, dx}{\overline{F}(t)} \leq \mu_1 \ .$$

4.3 Expected Time Between Failure Removals

If an item is replaced either at age t or at failure, whichever
occurs first, then the expected time between failure removals will be

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \ .$$

This ratio is nonincreasing in t if F is IHR . This ratio
also occurs in a bound on the renewal function. Suppose F is
IHR and $N(t)$ is the number of renewals in [0 , t] for an
associated renewal process. Then it can be shown ([5], p. 8) that

$$E[N(t)] \leq t \, F(t) / \int_0^t \overline{F}(x) \, dx \leq \frac{t}{\mu_1} \ .$$

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It is therefore of some interest to obtain bounds on this ratio.

THEOREM 4.4: If F is IHR, $F(0) = 0$ and $\mu_1 = 1$, μ_2 are specified, then

$$\text{(i)} \qquad \frac{\int_0^t \overline{F}(x)\, dx}{\overline{F}(t)} \geq \begin{cases} \inf_{G_T \in \mathscr{K}_3} \dfrac{\int_0^t \overline{G}_T(x)\, dx}{G_T(t)} & t \leq \mu_1 = 1 \\[2em] \inf_{G_T \in \mathscr{K}_3 \cup \mathscr{K}_4} \dfrac{\int_0^t \overline{G}_T(x)\, dx}{G_T(t)} & \mu_1 \leq t < T_1 \\[2em] \mu_1 & t \geq T_1 \end{cases}$$

$$\text{(ii)} \qquad \frac{\int_0^t \overline{F}(x)\, dx}{\overline{F}(t)} \leq \begin{cases} \infty & t \leq T_0 = 1 - \sqrt{\mu_2 - 1} \\[1em] \sup_{G_T \in \mathscr{K}_4} \dfrac{\int_0^t \overline{G}_T(x)\, dx}{G_T(t)} & T_0 \leq t \leq T_1 \\[1em] \sup_{G_T \in \mathscr{K}_3} \dfrac{\mu_1}{G_T(t)} & t > T_1 \end{cases}$$

All inequalities are sharp.

PROOF (i) Case 1. If $t \leq s(T_1)$, then $\overline{F}(x) \geq G_{T_1}(x)$ for $0 \leq x \leq t$ and

$$\int_0^t \overline{F}(x)\, dx \geq \int_0^t \overline{G}_{T_1}(x)\, dx$$

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implies the result.

Case 2. If $s(T_1) \leq t \leq s(\infty)$, then there exists T such that $F(t) = G_T(t)$ where $G_T \in \mathcal{H}_3$. This follows from the continuity of $s(T)$. Hence

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \geq \frac{\int_0^t \overline{G}_T(x) \, dx}{G_T(t)}$$

by (i) of Theorem 4.2. Since necessarily $s(\infty) > \mu_1$, the first bound in (i) is immediate.

Case 3. If $t \geq s(\infty)$ we use the function $v(T)$. Let $v(T)$ be the crossing in (T , ∞) from above of $1 - G_T$ by $1 - F$ if such a crossing exists. Otherwise let $v(T)$ be the right-hand endpoint of the support of F . Since $v(T)$ is continuous in T , choose T such that $v(T) = t$; i.e., we have
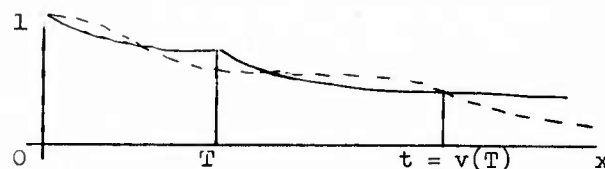
Since $G_T(t) = F(t)$ and clearly

$$\int_t^\infty \overline{F}(x) \, dx \leq \int_t^\infty \overline{G}(x) \, dx$$

we have

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \geq \frac{\int_0^t \overline{G}_T(x) \, dx}{G_T(t)} .$$

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Case 4. $t \geq T_1$. Since

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)}$$

is nonincreasing in t , we have

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \geq \mu_1$$

for all t . For $t > T_1$ this bound is attained by G_{T_1} . $\|$

(ii) Case 1. If $t \leq T_0$, then $F(t) \geq G_{T_0}(t) = 0$. Hence the inequality is trivial in this case.

Case 2. $t \leq u \, (T_0)$. We use the continuity of $u(T)$ again. If $t \leq u(T_0)$, then $\overline{F}(t) \leq \overline{G}_{T_0}(t)$ and

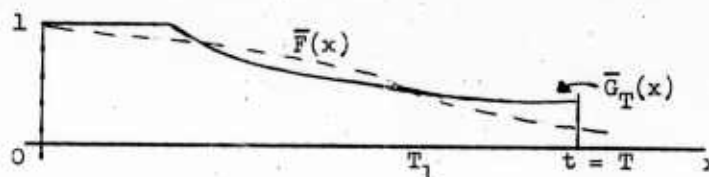$$\int_0^t \overline{F}(x) \, dx \leq \int_0^t \overline{G}_{T_0}(x) \, dx$$

implies the result.

Case 3. If $u(T_0) \leq t < u(T_1) = T_1$, then there exists T such that $G_T(t) = F(t)$ by continuity of $u(T)$. Note $G_T \in \mathcal{Y}_4$. By Theorem 4.2 (ii)

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \leq \frac{\int_0^t \overline{G}_T(x) \, dx}{G_T(t)} .$$

Case 4. $t \geq T_1$. There exists $G_T \in \mathcal{Y}_3$ corresponding to every t

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such $\overline{F}(t) \leq \overline{G}_T(t)$; i.e., $\overline{G}_T$ is the extremal distribution furnishing an upper bound on $\overline{F}(t)$ for $t \geq T_1$. The following drawing will help.

Since $\overline{F}(x) \geq \overline{G}_T(x) = 0$ for $x > T$ we have

$$\int_0^t \overline{F}(x) \, dx \quad \leq \quad \int_0^t \overline{G}_T(x) \, dx \, ,$$

and $F(t) \geq G_T(t)$ implies

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \quad \leq \quad \frac{\int_0^t \overline{G}_T(x) \, dx}{G_T(t)} \quad = \quad \frac{\mu_1}{G_T(t)} . \quad \|$$

$\overline{G}_T(t)$ is tabled for $\mu_1 = 1$ and μ_2 $(1 \leq \mu_2 \leq 2)$ as the upper bound on $\overline{F}(t)$ for the case $t \geq T_1$.

The asymptotic expected cost per unit time under an age replacement policy where an item is replaced either at failure or at time t whichever occurs first is [cf. [7] p. 68]

$$C(t) \quad = \quad \frac{(c_1 - c_2) \, F(t) + c_2}{\int_0^t \overline{F}(x) \, dx}$$

$(c_1 \geq c_2)$. The methods of Theorem 4.4 will also provide sharp

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bounds on this function.

If only μ_1 is specified, (F IHR), then the following inequalities are sharp:

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \geq \mu_1 \qquad \text{for all } t \, ,$$

$$\frac{\int_0^t \overline{F}(x) \, dx}{F(t)} \leq \begin{cases} \infty & t \leq \mu_1 \\ \dfrac{\mu_1}{1 - e^{-wt}} & t > \mu_1 \end{cases}$$

where w depends on t and is determined by

$$\int_0^t e^{-wx} \, dx = \mu_1 = 1 \, .$$

The first inequality is attained by the exponential. The second inequality is attained by the degenerate and exponentials truncated on the right.

4.4 Expected Time Between Planned Replacements

If an item is replaced either at time t or at failure, whichever occurs first, then the expected time between planned replacements will be

$$\frac{\int_0^t \overline{F}(x) \, dx}{\overline{F}(t)} \, .$$

This ratio is always nondecreasing in t .

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<u>THEOREM 4.5</u> If F is IHR , $F(0) = 0$ and $\mu_1 = 1$, μ_2 are specified, then

$$\frac{\int_0^t \overline{F}(x)\, dx}{\overline{F}(t)} \;\geq\; \left[\begin{array}{ll} t & 0 \leq t \leq T_0 \\[3ex] \displaystyle \inf_{\mathscr{D}_3 \cup \mathscr{D}_4} \frac{\int_0^t \overline{G}_T(x)\, dx}{\overline{G}_T(t)} & T_0 \leq t \leq T_1 \\[4ex] \displaystyle \inf_{\mathscr{D}_3} \frac{\int_0^t \overline{G}_T(x)\, dx}{\overline{G}_T(t)} & t > T_1 \end{array} \right.$$

$$\frac{\int_0^t \overline{F}(x)\, dx}{\overline{F}(t)} \;\leq\; \left[\begin{array}{ll} \displaystyle \sup_{\mathscr{D}_3} \frac{\int_0^t \overline{G}_T(x)\, dx}{\overline{G}_T(t)} & 0 \leq t \leq T_0 \\[4ex] \displaystyle \sup_{\mathscr{D}_3 \cup \mathscr{D}_4} \frac{\int_0^t \overline{G}_T(x)\, dx}{\overline{G}_T(t)} & T_0 < t \leq T_1 \\[4ex] \infty & t > T_1 \end{array} \right.$$

All inequalities are sharp.

<u>PROOF</u>: Clearly $\overline{F}(x) \geq \overline{F}(t)$ for $0 \leq x \leq t$ implies

$$\int_0^t \overline{F}(x)\, dx \;\geq\; \int_0^t \overline{F}(t)\, dx \;=\; t\,\overline{F}(t)$$

for all $t > 0$. For $t \leq T_0$, the lower bound is attained with $\overline{G}_{T_0}$.

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Now we consider cases, according as $t < s(T_1)$ or $t > s(T_1)$, where $s(T_1)$ denotes the crossing point of $\overline{G}_{T_1}(x)$ from above by $\overline{F}(x)$.

Case 1. $T_0 \leq t \leq s(T_1)$. Let $G_T \in \mathscr{Y}_4$ and let $a_1(T)$ denote a solution parameter for T . [see preliminaries to this section] It can be shown that $a_1(T)$ is a continuous function of T ([1] p. 52). Let $w(T)$ denote the crossing of $\overline{G}_T$ by $\overline{F}$ from above in $[0 , T]$ if it exists. Clearly $w(T_1) = s(T_1)$ and

$$0 \leq w(T) \leq s(T_1) .$$

Continuity of $a_1(T)$ together with log concavity of $\overline{F}(x)$ insures that for t satisfying $T_0 \leq t \leq s(T_1)$ there exists T such that $w(T) = t$ where $G_T \in \mathscr{Y}_4$. A drawing will help:
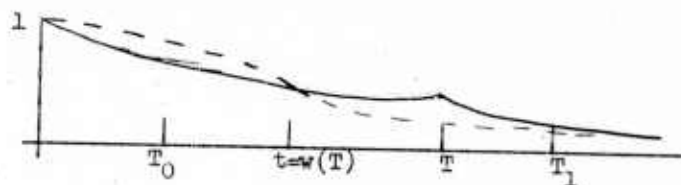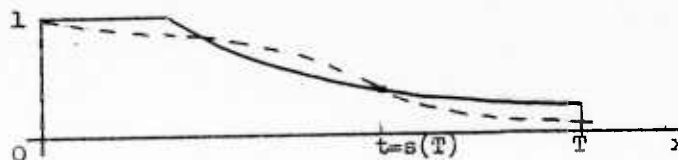

Clearly we will have $\overline{F}(x) \geq \overline{G}_T(x)$ for $x \leq t$ and hence

$$\frac{\int_0^t \overline{F}(x) \ dx}{\overline{F}(t)} \geq \frac{\int_0^t \overline{G}_T(x) \ dx}{\overline{G}_T(t)} .$$

Case 2. $s(T_1) < t < s(\infty)$.

Choose T and $G_T \in \mathscr{Y}_3$ such that $s(T) = t$; i.e., we will have

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This is possible by the continuity of $s(T)$. Hence

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \geq \frac{\int_0^t \overline{G}_T(x)\ dx}{\overline{G}_T(t)}$$

by Theorem 4.2. Note that $s(T_1) \leq T_1$ so that we need only consider $G_T \in \mathcal{A}_3$ for $t \geq T_1$.

Case 3. $t > s(\infty)$

In this case $\overline{F}(x) \leq \overline{G}_{T_0}(x)$ for $x \geq t$ and therefore

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \geq \frac{\int_0^t \overline{G}_{T_0}(x)\ dx}{\overline{G}_{T_0}(t)} \ . \ \|$$

To prove the upper bound we again take cases.

Case 1. $0 \leq t \leq T_0$. We may suppose $\overline{F}(t) < 1$ $(0 \leq t \leq T_0)$ since otherwise
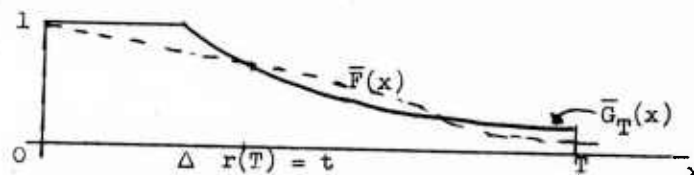
$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} = t$$

and this value is realized by $G_{T_0} \in \mathcal{A}_3$. Hence this is considered when we take supremums over $\mathcal{A}_3$. We use crucially the continuity

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of $r(T)$ where $r(T)$ is the point in (Δ , T) that $\overline{F}$ crosses $\overline{G}_T$ from below if such a crossing exists. Clearly $r(T_0) \geq T_0$; i.e.,
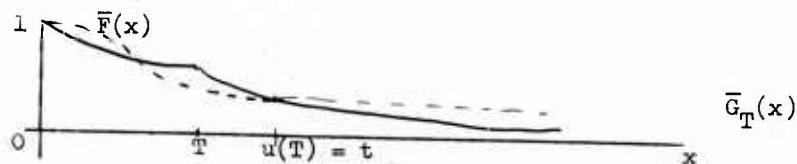

Consider now $0 \leq t \leq r(T_0)$. By continuity of $r(T)$ (see [1]) we can choose T so that $r(T) = t$. Then we have

By definition of $r(T)$, $\overline{F}(x) \leq \overline{G}_T(x)$ for $0 \leq x \leq t$ and $F(t) = G_T(t)$ where $G_T \in \mathscr{D}_3$. Hence

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \leq \frac{\int_0^t \overline{G}_T(x)\ dx}{\overline{G}_T(t)}$$

Case 2. $T_0 \leq t \leq T_1$. By case 1, we may assume t satisfies $r(T_0) = u(T_0) \leq t \leq u(T_1) = T_1$. By continuity of $u(T)$ choose T and $G_T \in \mathscr{D}_4$ such that $u(T) = t$. The following drawing will help;

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By Case (3) of (ii) of Theorem 4.2 we have $F(t) = G_T(t)$ and

$$\int_0^t \overline{F}(x)\ dx \ \leq \ \int_0^t \overline{G}_T(x)\ dx$$

hence

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \ \leq \ \frac{\int_0^t \overline{G}_T(x)\ dx}{\overline{G}_T(t)} \ .$$

Case 3. $t > T_1$. In this case the upper bound is attained by $G_{T_1}(t)$; i.e.,

$$\frac{\int_0^t \overline{G}_{T_1}(x)\ dx}{\overline{G}_{T_1}(t)} \ = \ \mu_1/0 = +\infty \ . \ \|$$

If the only moment specified is the mean μ_1 (F IHR), then the following inequalities are sharp.

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \ \geq \ \left[\begin{array}{ll} t & t \leq \mu_1 \\[3em] \dfrac{\inf\limits_{w} \int_0^t \overline{G}_w(x)\ dx}{\overline{G}_w(t)} & t > \mu_1 \end{array} \right.$$

$$\frac{\int_0^t \overline{F}(x)\ dx}{\overline{F}(t)} \ \leq \ \left[\begin{array}{ll} \dfrac{\sup\limits_{\Delta} \int_0^t \overline{G}_\Delta(x)\ dx}{\overline{G}_\Delta(t)} & t < \mu_1 \\[3em] + \infty & t \geq \mu_1 \end{array} \right.$$

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where

$$\overline{G}_w(x) = \begin{cases} e^{-wx} & x < T \\ 0 & x \geq T \end{cases}$$

w   and   T   are chosen to satisfy

$$\int_0^T e^{-wx} \, dx = \mu_1$$

and

$$\overline{G}_\Delta(x) = \begin{cases} 1 & 0 \leq x \leq \Delta \\ e^{-\frac{x - \Delta}{\mu_1 - \Delta}} & x \geq \Delta \end{cases}$$

$(0 \leq \Delta \leq \mu_1)$. The proof is similar to previous proofs and uses the methods of [1] pp. 43-44.

Acknowledgment   This paper represents an extension of part of a previous paper [1] with Albert Marshall to whom I would like to acknowledge a great indebtedness.

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